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B.Sc - part-I (H), paper - II Date - 15-04-2020

Analytical Geometry of two dimensions.

Q.1 Find the centre, the length of the axes and the eccentricity of the ellipse $2x^2+3y^2-4x-12y+13=0$.

Soln! The given equation is

$$2x^2+3y^2-4x-12y+13=0$$

$$\Rightarrow (2x^2-4x)+(3y^2-12y)+13=0$$

$$\Rightarrow 2(x^2-2x)+3(y^2-4y)+13=0$$

$$\Rightarrow 2\left[(x-1)^2-1\right]+3\left[(y-2)^2-4\right]+13=0$$

$$\Rightarrow 2(x-1)^2+3(y-2)^2=1$$

$$\Rightarrow \frac{(x-1)^2}{\frac{1}{2}}+\frac{(y-2)^2}{\frac{1}{3}}=1$$

Therefore, the centre is the point $(1,2)$.

Let $a^2=\frac{1}{2}$ and $b^2=\frac{1}{3}$, where $2a$ and $2b$ are the lengths of the major and minor axes respectively.

Now, $a=\frac{1}{\sqrt{2}}$, $\therefore 2a=2\times\frac{1}{\sqrt{2}}=\frac{2}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}=\frac{2\sqrt{2}}{2}=\sqrt{2}$.

and $b=\frac{1}{\sqrt{3}}$, $2b=2\times\frac{1}{\sqrt{3}}=\frac{2}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$

Thus, the lengths of the axes are $\sqrt{2}$ and $\frac{2\sqrt{3}}{3}$.

Again, $b^2=a^2(1-e^2)\Rightarrow\left(\frac{1}{\sqrt{3}}\right)^2=\left(\frac{1}{\sqrt{2}}\right)^2(1-e^2)$

$$\Rightarrow \frac{1}{3}=\frac{1}{2}(1-e^2)$$

$$\Rightarrow 2=3-3e^2 \Rightarrow 3e^2=3-2$$

$$\Rightarrow e^2=\frac{1}{\sqrt{3}}$$

$$\Rightarrow e=\frac{1}{\sqrt{3}}=\frac{1}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}=\frac{\sqrt{3}}{\sqrt{3}}.$$

Hence, the centre is the point $(1,2)$.
the length of the axes are $\sqrt{2}$ and $\frac{2\sqrt{3}}{3}$
the eccentricity is $\frac{\sqrt{3}}{3}$ Ans

Q.2. Find the equation of the hyperbola in standard form whose eccentricity is $\sqrt{2}$ and the distance between whose foci is 16.

Sol^n: Let e be the eccentricity, then $e = \sqrt{2}$

Also, if S and S' be the foci, then $S'S = 16$

But $S'S = CS' + CS$, where C is the centre of the hyperbola

But $CS = CS' = ae$ which is magnitude

$\therefore S'S = 2ae$ and we have $2ae = 16$

Thus, we have $2a \times \sqrt{2} = 16 \Rightarrow a = \frac{16}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{16\sqrt{2}}{4} = 4\sqrt{2}$.

Also, $b^2 = a^2(e^2-1) = (4\sqrt{2})^2\{(\sqrt{2})^2-1\} = 16 \times 2\,(2-1) = 32$

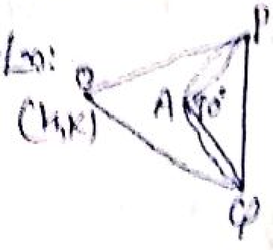
The equation to the hyperbola is therefore

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{32} - \frac{y^2}{32} = 1 \Rightarrow x^2 - y^2 = 32$$

This is the required eq^n.

Q.3. A chord of the parabola $y^2 = 4ax$ subtends a right angle at the vertex. Prove that the tangent at the extremities of the chord intersect on the line $x + 4a = 0$.

Sol^n:



Let us suppose that the tangents at the extremities P and Q meet at the point O(h,k). Equation to the chord PQ is

$$yk = 2a(x+h) \quad \text{——— (1)}$$

Since, A(0,0) is the vertex of the parabola.

The equation to the pair of st. lines AP and AQ both passing through the origin and at right angles to each other.

By (1), $\frac{ky - 2ax}{2ah} = 1$. We make the eq^n $y^2 = 4ax$ a homogeneous equation of second degree in x and y

Thus $y^2 = 4ax\left\{\frac{ky - 2ax}{2ah}\right\} \Rightarrow y^2 \times 2ah = 4akxy - 8a^2x^2$

$\Rightarrow 4ax^2 - 2kxy + hy^2 = 0$.

This equation represents a pair of straight lines ⊥ to each other.

Hence, co-efficient of x^2 + co-efficient of $y^2 = 0 \Rightarrow 4a + h = 0$

Thus, the locus of (h,k) is $x + 4a = 0$

Therefore, the tangents at the extremities P and Q of the chord intersect on the line $x + 4a = 0$

Proved.

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B.Sc. part-III(H), paper-III
Differential Calculus: Date- 15-04-2020.

Q.1: If $y = \sin^{-1}x$, prove that
$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$

Soln: We have $y = \sin^{-1}x$
Differentiating both sides with respect to x, we get
$$y_1 = \frac{1}{\sqrt{1-x^2}} \Rightarrow \sqrt{1-x^2}\,y_1 = 1$$
Again, differentiating both sides with respect to x,
we get, $\sqrt{1-x^2}\,y_2 + \frac{1}{2\sqrt{1-x^2}}(-2x)y_1 = 0$
$$\Rightarrow (1-x^2)y_2 - xy_1 = 0.$$
By Leibnitz's theorem to differentiate n times with respect to x, we get
$$y_{n+2}(1-x^2) + \frac{n}{1}y_{n+1}(-2x) + \frac{n(n-1)}{1\cdot 2}y_n(-2) - y_{n+1}x - \frac{n}{1}y_n = 0$$
$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$$
Proved.

Q.2. If $y = \tan^{-1}x$, prove that $(1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0$.

Soln: $\because y = \tan^{-1}x$
Differentiating w.r. to x, we get
$$y_1 = \frac{1}{1+x^2} \Rightarrow y_1(1+x^2) = 1$$
Differentiating n times w.r. to x, by Leibnitz's theorem,
$$y_{n+1}(1+x^2) + {}^nC_1 y_n 2x + {}^nC_2 y_{n-1} 2 = 0$$
$$\Rightarrow (1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0.$$
Proved.

B.sc-part-II(H), paper-III, Date-15-04-2020.

Q:3. If $y = \frac{1}{x^2+a^2}$ find y_n.

Soln: We have, $y = \frac{1}{x^2+a^2} = \frac{1}{x^2-(-1)a^2} = \frac{1}{x^2-i^2a^2} = \frac{1}{(x+ia)(x-ia)}$

$$\Rightarrow y = \frac{1}{2ia}\left[\frac{(x+ia)-(x-ia)}{(x-ia)(x+ia)}\right]$$

$$= \frac{1}{2ia}\left[\frac{1}{x-ia} - \frac{1}{x+ia}\right]$$

Differentiating both sides n times with respect to x, we get

$$y_n = \frac{1}{2ia}\left[\frac{(-1)^n \lfloor n}{(x-ia)^{n+1}} - \frac{(-1)^n \lfloor n}{(x+ia)^{n+1}}\right]$$

$$\Rightarrow y_n = \frac{(-1)^n \lfloor n}{2ia}\left[\frac{1}{(x-ia)^{n+1}} - \frac{1}{(x+ia)^{n+1}}\right]$$

Putting $x = r\cos\theta$ and $y = r\sin\theta$

Squaring and adding above two, we get

$x^2+a^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta+\sin^2\theta) = r^2.1 = r^2$,

and also dividing them, we get

$$\frac{x}{a} = \frac{r\cos\theta}{r\sin\theta} \Rightarrow \frac{x}{a} = \cot\theta \Rightarrow \theta = \cot^{-1}\frac{x}{a}$$

$$\therefore y_n = \frac{(-1)^n \lfloor n}{2ia}\left[\frac{1}{(r\cos\theta - ir\sin\theta)^{n+1}} - \frac{1}{(r\cos\theta + ir\sin\theta)^{n+1}}\right]$$

$$= \frac{(-1)^n \lfloor n}{2ia}\left[\frac{1}{r^{n+1}(\cos\theta - i\sin\theta)^{n+1}} - \frac{1}{r^{n+1}(\cos\theta + i\sin\theta)^{n+1}}\right]$$

$$= \frac{(-1)^n \lfloor n}{2ia\, r^{n+1}}\left[(\cos\theta - i\sin\theta)^{-(n+1)} - (\cos\theta + i\sin\theta)^{-(n+1)}\right]$$

$$= \frac{(-1)^n \lfloor n}{2ia\, r^{n+1}}\left[\cos(n+1)\theta + i\sin(n+1)\theta - \cos(n+1)\theta + i\sin(n+1)\theta\right]$$

$$= \frac{(-1)^n \lfloor n}{\cancel{2ia}\cdot r^{n+1}} \cancel{2i}\sin(n+1)\theta \Rightarrow y_n = \frac{(-1)^n \lfloor n}{a\left(\frac{a}{\sin\theta}\right)^{n+1}} \sin(n+1)\theta$$

Hence, $y_n = \frac{(-1)^n \lfloor n}{a^{n+2}} \sin^{n+1}\theta \sin(n+1)\theta$, where $\theta = \cot^{-1}\frac{x}{a}$

solved